

# Turán numbers of extensions of some sparse hypergraphs via Lagrangians

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## Abstract

Given a positive integer  $n$  and an  $r$ -uniform hypergraph (or  $r$ -graph for short)  $F$ , the *Turán number*  $ex(n, F)$  of  $F$  is the maximum number of edges in an  $r$ -graph on  $n$  vertices that does not contain  $F$  as a subgraph. The *extension*  $H^F$  of  $F$  is obtained as follows: For each pair of vertices  $v_i, v_j$  in  $F$  not contained in an edge of  $F$ , we add a set  $B_{ij}$  of  $r - 2$  new vertices and the edge  $\{v_i, v_j\} \cup B_{ij}$ , where the  $B_{ij}$ 's are pairwise disjoint over all such pairs  $\{i, j\}$ . Let  $K_p^r$  denote the complete  $r$ -graph on  $p$  vertices. For all sufficiently large  $n$ , we determine the Turán numbers of the extensions of a 3-uniform  $t$ -matching, a 3-uniform linear star of size  $t$ , and a 4-uniform linear star of size  $t$ , respectively. We also show that the unique extremal hypergraphs are balanced blowups of  $K_{3t-1}^3$ ,  $K_{2t}^3$ , and  $K_{3t}^4$ , respectively. Our results generalize the recent result of Hefetz and Keevash [7].

Key Words: Turán number, Hypergraph Lagrangian

## 1 Notations and definitions

For a set  $V$  and a positive integer  $r$  we denote by  $V^{(r)}$  the family of all  $r$ -subsets of  $V$ . An  $r$ -uniform graph or  $r$ -graph  $G$  consists of a set  $V(G)$  of vertices and a set  $E(G) \subseteq V(G)^{(r)}$  of edges. When there is no confusion, we simply write  $G$  for  $E(G)$ . Let  $|G|$  denote the number of edges of  $G$ . An edge  $e = \{a_1, a_2, \dots, a_r\}$  will be simply denoted by  $a_1 a_2 \dots a_r$ . An  $r$ -graph  $H$  is a *subgraph* of an  $r$ -graph  $G$ , denoted by  $H \subseteq G$ , if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ . A subgraph of  $G$  *induced* by  $V' \subseteq V$ , denoted as  $G[V']$ , is the  $r$ -graph with vertex set  $V'$  and edge set  $E' = \{e \in E(G) : e \subseteq V'\}$ . Let  $K_t^r$  denote the *complete  $r$ -graph* on  $t$  vertices, that is, the  $r$ -graph on  $t$  vertices containing all  $r$ -subsets of the vertex set as edges. Let  $T_m^r(n)$  be the balanced blow-up of  $K_m^r$  on  $n$  vertices, i.e.,  $V(T_m^r(n)) = V_1 \cup V_2 \cup \dots \cup V_m$  such that  $V_i \cap V_j = \emptyset$  for every  $1 \leq i < j \leq m$  and  $|V_1| \leq |V_2| \leq \dots \leq |V_m| \leq |V_1| + 1$ , and  $E(T_m^r(n)) = \{S \in \binom{[n]}{r} : \forall i \in [m], |S \cap V_i| \leq 1\}$ . The graph  $T_m^r(n)$  is also commonly called *the  $r$ -uniform*

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$m$ -partite Turán graph on  $n$  vertices. Let  $t_m^r(n) = |T_m^r(n)|$ . For a positive integer  $n$ , we let  $[n]$  denote  $\{1, 2, 3, \dots, n\}$ . Given positive integers  $m$  and  $r$ , let  $[m]_r = m(m-1)\dots(m-r+1)$ .

Given an  $r$ -graph  $F$ , an  $r$ -graph  $G$  is called  $F$ -free if it does not contain  $F$  as a subgraph. For a fixed positive integer  $n$  and an  $r$ -graph  $F$ , the Turán number of  $F$ , denoted by  $ex(n, F)$ , is the maximum number of edges in an  $r$ -graph on  $n$  vertices that does not contain  $F$  as a subgraph. An averaging argument of Katona, Nemetz and Simonovits [10] shows that the sequence  $\frac{ex(n, F)}{\binom{n}{r}}$  is a non-increasing sequence of real numbers in  $[0, 1]$ . Hence,  $\lim_{n \rightarrow \infty} \frac{ex(n, F)}{\binom{n}{r}}$  exists. The Turán density of  $F$  is defined as

$$\pi(F) = \lim_{n \rightarrow \infty} \frac{ex(n, F)}{\binom{n}{r}}.$$

In this paper, we extend the work of Hefetz and Keevash in [7] and determine Turán numbers of several classes of  $r$ -graphs using so-called hypergraph Lagrangian method.

**Definition 1.1** Let  $G$  be an  $r$ -graph on  $[n]$  and let  $\vec{x} = (x_1, \dots, x_n) \in [0, \infty)^n$ . For every subgraph  $H \subseteq G$ , define

$$\lambda(H, \vec{x}) = \sum_{e \in E(H)} \prod_{i \in e} x_i.$$

The Lagrangian of  $G$ , denoted by  $\lambda(G)$ , is defined as

$$\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in \Delta\},$$

where

$$\Delta = \{\vec{x} = (x_1, x_2, \dots, x_n) \in [0, \infty)^n : \sum_{i=1}^n x_i = 1\}.$$

The value  $x_i$  is called the *weight* of the vertex  $i$  and a vector  $\vec{x} \in \Delta$  is called a *feasible weight vector* on  $G$ . A feasible vector  $\vec{y} \in \Delta$  is called an *optimum weight vector* on  $G$  if  $\lambda(G, \vec{y}) = \lambda(G)$ .

Given an  $r$ -graph  $F$ , we define the *Lagrangian density*  $\pi_\lambda(F)$  of  $F$  to be

$$\pi_\lambda(F) = \sup\{r!\lambda(G) : F \not\subseteq G\}.$$

**Proposition 1.2**  $\pi(F) \leq \pi_\lambda(F)$ . ■

*Proof.* Let  $\varepsilon > 0$  be arbitrary. Let  $n$  be large enough and let  $G_n$  be a maximum  $F$ -free  $r$ -graph on  $n$  vertices. We have

$$\pi(F) \leq \frac{|G_n|}{\binom{n}{r}} + \varepsilon/2 \leq r! \sum_{e \in E(G_n)} \frac{1}{n^r} + \varepsilon = r!\lambda(G_n, (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})) + \varepsilon \leq r!\lambda(G_n) + \varepsilon \leq \pi_\lambda(F) + \varepsilon.$$

■

The Lagrangian method for hypergraph Turán problems were developed independently by Sidorenko [18] and Frankl-Füredi [5], generalizing work of Motzkin and Straus [12] and Zykov [23]. More recent developments of the method were obtained by Pikhurko [16] and Norin and Yepremyan [14]. Based on these developments, Brandt, Irwin, and Jiang [2], and independently Norin and Yepremyan [15] were able to determine the Turán numbers of a large family of hypergraphs and thereby extending earlier

works in [1, 4, 8, 9, 17, 19]. The methods used by the two groups are quite different. The former group used Pikhurko's stability method while the latter group used a refined stability method that they developed in [14]. In this paper, we extend a recent work on the topic by Hefetz and Keevash [7] on Lagrangians of intersecting 3-graphs to determine the maximum Lagrangian of a 3-graph not containing a matching of a given size. We also determine the maximum Lagrangian of a 3-graph not containing a linear star of a given size and the maximum Lagrangian of a 4-graph not containing a linear star of a given size. These results combined with the corresponding general theorems in [2] and [15] then allow us to determine the Turán numbers of some corresponding hypergraphs, which we now define as below.

We say that a pair of vertices  $\{i, j\}$  is *covered* in a hypergraph  $H$  if there exists  $e \in H$  such that  $\{i, j\} \subseteq e$ . Let  $r \geq 3$  and  $F$  be an  $r$ -graph. Let  $p \geq |V(F)|$ . Let  $\mathcal{K}_p^F$  denote the family of  $r$ -graphs  $H$  that contains a set  $C$  of  $p$  vertices, called the *core*, such that the subgraph of  $H$  induced by  $C$  contains a copy of  $F$  and such that every pair in  $C$  that are not covered by  $F$  is covered by an edge of  $H$ . We call  $\mathcal{K}_p^F$  the family of *weak extensions* of  $F$  for the given  $p$ . If  $p = |V(F)|$ , then we simply call  $\mathcal{K}_p^F$  the family of *extensions* of  $F$ . Let  $H_p^F$  be a member of  $\mathcal{K}_p^F$  obtained as follows. Label the vertices of  $F$  as  $v_1, \dots, v_{|V(F)|}$ . Add new vertices  $v_{|V(F)|+1}, \dots, v_p$ . Let  $C = \{v_1, \dots, v_p\}$ . For each pair of vertices  $v_i, v_j \in C$  not covered in  $F$ , we add a set  $B_{ij}$  of  $r - 2$  new vertices and the edge  $\{v_i, v_j\} \cup B_{ij}$ , where the  $B_{ij}$ 's are pairwise disjoint over all such pairs  $\{i, j\}$ . We call  $H_p^F$  the *extension* of  $F$  for the given  $p$ . If  $p = |V(F)|$ , then we simply call  $H_p^F$  the *extension* of  $F$ .

Let  $r, t$  be integers such that  $r \geq 3$  and  $t \geq 2$ . The  $r$ -uniform  $t$ -matching, denoted by  $M_t^r$ , is the  $r$ -graph with  $t$  pairwise disjoint edges. The  $r$ -uniform linear star of size  $t$ , denoted by  $L_t^r$ , is the  $r$ -graph with  $t$  edges such that these  $t$  edges contain a common vertex  $x$  but are pairwise disjoint outside  $\{x\}$ .

In [7], Hefetz and Keevash determined the Lagrangian density of  $M_2^3$  and the Turán number of the extension of  $M_2^3$  for all sufficiently large  $n$ . In this paper, we generalize their result to determine the Lagrangian density of  $M_t^3$  for all  $t \geq 2$ . We also determine the Lagrangian densities of  $L_t^3$  or  $L_t^4$ , for all  $t \geq 2$ . For each of the hypergraphs mentioned above, we determine the Turán numbers of their extensions for all sufficiently large  $n$ . Our method differs from the one employed by Hefetz and Keevash [7]. For the matching problem, we use compression and induction. This allows us to obtain a short proof of the main result of [7] and solve the problem for general  $t$ . We solve the linear star problem for  $r = 3, 4$  by first studying a local version of the matching problem for  $r = 2, 3$ , respectively.

## 2 Preliminaries

In this section, we develop some useful properties of Lagrangian functions. The following fact follows immediately from the definition of the Lagrangian.

**Fact 2.1** *Let  $G_1, G_2$  be  $r$ -graphs and  $G_1 \subseteq G_2$ . Then  $\lambda(G_1) \leq \lambda(G_2)$ .*

Given an  $r$ -graph  $G$  and a set  $S$  of vertices, the *link graph* of  $S$  in  $G$ , denoted by  $L_G(S)$ , is the hypergraph with edge set  $\{e \in \binom{V(G) \setminus S}{r-|S|} : e \cup S \in E(G)\}$ . When  $S$  has only one element, e.g.  $S = \{i\}$ , we write  $L_G(i)$  for  $L_G(\{i\})$ . Furthermore, when there is no confusion, we will drop the subscript  $G$ . Given  $i, j \in V(G)$ , define

$$L_G(j \setminus i) = \left\{ f \in \binom{V(G) \setminus \{i, j\}}{r-1} : f \cup \{j\} \in E(G) \text{ and } f \cup \{i\} \notin E(G) \right\},$$

and define

$$\pi_{ij}(G) = (E(G) \setminus \{f \cup \{j\} : f \in L_G(j \setminus i)\}) \cup \{f \cup \{i\} : f \in L_G(j \setminus i)\}.$$

By the definition of  $\pi_{ij}(G)$ , it's straightforward to verify the following fact.

**Fact 2.2** *Let  $G$  be an  $r$ -graph on the vertex set  $[n]$ . Let  $\vec{x} = (x_1, x_2, \dots, x_n)$  be a feasible weight vector on  $G$ . If  $x_i \geq x_j$ , then  $\lambda(\pi_{ij}(G), \vec{x}) \geq \lambda(G, \vec{x})$ .*

Part (a) of the following lemma is well-known (see [3] for instance). We include a short proof of it for completeness.

**Lemma 2.3** *Let  $r, t \geq 2$  be integers. Let  $G$  be a  $M_t^r$ -free  $r$ -graph on the vertex set  $[n]$ . Let  $i, j$  be a pair of vertices, then the following hold:*

(a)  $\pi_{ij}(G)$  is  $M_t^r$ -free.

(b) If  $G$  is also  $K_{tr-1}^r$ -free and  $\{i, j\}$  is contained in an edge of  $G$ , then  $\pi_{ij}(G)$  is  $K_{tr-1}^r$ -free.

*Proof.* Suppose for contradiction that there exist  $i, j$  such that  $\pi_{ij}(G)$  contains a  $t$ -matching  $M$ . Then there must be an edge  $e$  of  $M$  that is in  $\pi_{ij}(G)$  but not in  $G$ . This implies that  $i \in e$ ,  $j \notin e$  and  $e' = (e \setminus \{i\}) \cup \{j\} \in G$ . If  $j$  is not covered by any edge of  $M$ , then  $(M \setminus \{e\}) \cup \{e'\}$  is a  $t$ -matching in  $G$ , contradicting  $G$  being  $M_t^r$ -free. Hence,  $\exists f \in M$  such that  $j \in f$ . Let  $f' = (f \setminus \{j\}) \cup \{i\}$ . By the definition of  $\pi_{ij}(G)$ ,  $f$  and  $f'$  must both exist in  $G$ , or else  $f$  wouldn't be in  $\pi_{ij}(G)$ . But now,  $(M \setminus \{e, f\}) \cup \{e', f'\}$  is a  $t$ -matching in  $G$ , contradicting  $G$  being  $M_t^r$ -free.

Next, suppose that  $G$  is  $K_{tr-1}^r$ -free and  $\{i, j\}$  is contained in some edge  $e$  of  $G$ . Suppose for contradiction that  $\pi_{ij}(G)$  contains a copy  $K$  of  $K_{tr-1}^r$ . Clearly  $V(K)$  must contain  $i$ . If  $V(K)$  also contains  $j$  then it is easy to see that  $K$  also exists in  $G$ , contradicting  $G$  being  $K_{tr-1}^r$ -free. All the edges in  $K$  not containing  $i$  also exist in  $G$ . By our assumption,  $V(K)$  contains at least  $tr - 1 - (r - 1) = (t - 1)r$  vertices outside  $e$ . So  $K$  contains a  $(t - 1)$ -matching  $M$  disjoint from  $e$ , all of which lie in  $G$  by earliest discussion. Now,  $M \cup \{e\}$  is a  $t$ -matching in  $G$ , a contradiction. ■

Next, we show that for  $r = 2$ , part (b) of Lemma 2.3 holds even without the assumption that  $\{i, j\}$  is contained in an edge.

**Lemma 2.4** *Let  $t \geq 2$ . Let  $G$  be an  $M_t^2$ -free and  $K_{2t-1}^2$ -free graph on  $[n]$  and  $i, j \in [n]$ . Then  $\pi_{ij}(G)$  is also  $K_{2t-1}^2$ -free.*

*Proof.* Suppose for contradiction that  $\pi_{ij}(G)$  contains a copy  $K$  of  $K_{2t-1}^2$ . Then  $\pi_{ij}(G) \neq G$  and  $K$  contains  $i$  but not  $j$  (note that  $\pi_{ij}$  does not change the common link of  $i$  and  $j$ ). Since  $\pi_{ij}(G) \neq G$ ,  $L_G(j \setminus i) \neq \emptyset$ . Also,  $L_G(i \setminus j) \neq \emptyset$ , since otherwise  $K \subseteq G$ . Let  $a \in V(L_G(i \setminus j))$ ,  $b \in V(L_G(j \setminus i))$ . Note that any edge in  $\pi_{ij}(G)$  not containing  $i$  also exist in  $G$ . Hence,  $K - \{i, a, b\}$  is a complete graph on at least  $2t - 4$  vertices in  $G$ , which contains a  $(t - 2)$ -matching  $M$ . Now,  $M \cup \{ia, jb\}$  is a  $t$ -matching in  $G$ , a contradiction. ■

An  $r$ -graph  $G$  is *dense* if for every subgraph  $G'$  of  $G$  with  $|V(G')| < |V(G)|$  we have  $\lambda(G') < \lambda(G)$ . This is equivalent to saying that all optimum weight vectors on  $G$  are in the interior of  $\Delta$ , which means that no coordinate in an optimum weight vector is zero. We say that a hypergraph  $G$  *covers pairs* if every pair of its vertices is covered by an edge.

**Fact 2.5** ([6]) *Let  $G = (V, E)$  be a dense  $r$ -graph. Then  $G$  covers pairs.*

**Definition 2.6** Let  $G$  be an  $r$ -graph on  $[n]$  and a linear order  $\mu$  on  $[n]$ . We say that  $G$  is *left-compressed* (or simply *compressed*) relative to  $\mu$  if for all  $i, j \in [n]$  with  $i <_\mu j$  we have  $\pi_{ij}(G) = G$ . Let  $\vec{x}$  be a feasible weight vector on  $G$ . We say that  $G$  is  $\vec{x}$ -*compressed* if there exists a linear order  $\mu$  on  $V(G)$  such that  $\forall i, j \in V(G)$  whenever  $i <_\mu j$  we have  $x_i \geq x_j$  and that  $G$  is left-compressed relative to  $\mu$ .

**Algorithm 2.7** Let  $G$  be an  $r$ -graph on  $[n]$ . Let  $\vec{x}$  be an optimum weight vector of  $G$ . If there exist vertices  $i, j$ , where  $i < j$ , such that  $x_i > x_j$  and  $L_G(j \setminus i) \neq \emptyset$ , then replace  $G$  by  $\pi_{ij}(G)$ , continue this process until no such pair exists.

In the above algorithm, by relabelling the vertices if necessary, we may assume that  $x_1 \geq x_2 \geq \dots \geq x_n$ . Note that  $s(G) = \sum_{e \in G} \sum_{i \in e} i$  is a positive integer that decreases by at least 1 in each step. Hence the algorithm terminates after finite many steps.

**Algorithm 2.8** (Dense and compressed subgraph)

**Input:** An  $r$ -graph  $G$ .

**Output:** A dense subgraph  $G' \subseteq G$  together with an optimum weight vector  $\vec{y}$  such that  $\lambda(G', \vec{y}) = \lambda(G)$  and that  $G'$  is  $\vec{y}$ -compressed.

**Step 1.** If  $G$  is not dense, then replace  $G$  by a dense subgraph with the same Lagrangian. Otherwise, go to Step 2.

**Step 2.** Let  $\vec{y}$  be an optimum weight vector of  $G$ . If  $G$  is  $\vec{y}$ -compressed, then terminate. Otherwise, there exist vertices  $i, j$ , where  $i < j$ , such that  $y_i > y_j$  and  $L_G(j \setminus i) \neq \emptyset$ , then replace  $G$  by  $\pi_{ij}(G)$  and go to step 1.

Note that the algorithm terminates after finite many steps since Step 1 reduces the number of vertices by at least 1 in each step and Step 2 reduces the parameter  $s(G)$  (similarly defined as above) by at least 1 in each step.

**Lemma 2.9** *Let  $G$  be a  $M_t^r$ -free  $r$ -graph and  $\vec{x}$  a feasible weight vector on  $G$ . Then the following hold:*

(a) *There exists a  $M_t^r$ -free  $r$ -graph  $H$  with  $V(H) = V(G)$  such that  $\lambda(H, \vec{x}) \geq \lambda(G, \vec{x})$  and that  $H$  is  $\vec{x}$ -compressed.*

(b) *There exists a dense  $M_t^r$ -free  $r$ -graph  $G'$  with  $|V(G')| \leq |V(G)|$  together with an optimum weight vector  $\vec{y}$  such that  $\lambda(G', \vec{y}) = \lambda(G)$  and that  $G'$  is  $\vec{y}$ -compressed. Furthermore, if  $G$  is  $K_{tr-1}^r$ -free, then  $G'$  is  $K_{tr-1}^r$ -free.*

*Proof.* For (a), we apply Algorithm 2.7 to  $G$  and let  $H$  be the final graph obtained. That  $\lambda(H, \vec{x}) \geq \lambda(G, \vec{x})$  follows from Fact 2.2. That  $H$  is  $M_t^r$ -free follows from Lemma 2.3. That  $H$  is  $\vec{x}$ -compressed follows from the fact that algorithm terminates after finite steps and it only terminates when the  $r$ -graph becomes compressed.

For (b), we apply Algorithm 2.8 to  $G$  and let  $G'$  be the final graph and  $\vec{y}$  the optimum weight vector on  $G$  implied by the algorithm. Since Algorithm terminates after finite many steps,  $G'$  and  $\vec{y}$  are well-defined. By Fact 2.2,  $\lambda(G') \geq \lambda(G)$ . By Lemma 2.3,  $G'$  is  $M_t^r$ -free. By the algorithm,  $G'$  is  $\vec{y}$ -compressed. Assume that  $G$  is  $K_{tr-1}^r$ -free. In the process of obtaining  $G'$  we always take a dense subgraph first before applying a compression  $\pi_{ij}$ . Taking a subgraph preserves  $K_{tr-1}^r$ -free condition.

For a dense graph, by Lemma 2.3 part (b) performing  $\pi_{ij}$  preserves  $K_{tr-1}^T$ -free condition. So  $G'$  is  $K_{tr-1}^T$ -free. ■

In [12], Motzkin and Straus determined the Lagrangian of any given 2-graph.

**Theorem 2.10** (Motzkin and Straus [12]) *If  $G$  is a 2-graph in which a maximum complete subgraph has  $t$  vertices, then  $\lambda(G) = \lambda(K_t^2) = \frac{1}{2}(1 - \frac{1}{t})$ .* ■

Let  $G$  be an  $r$ -graph on  $[n]$  and  $\vec{x} = (x_1, x_2, \dots, x_n)$  be a weight vector on  $G$ . If we view  $\lambda(G, \vec{x})$  as a function in variables  $x_1, \dots, x_n$ , then

$$\frac{\partial \lambda(G, \vec{x})}{\partial x_i} = \sum_{i \in e \in E(G)} \prod_{j \in e \setminus \{i\}} x_j.$$

We sometimes write  $\frac{\partial \lambda}{\partial x_i}$  for  $\frac{\partial \lambda(G, \vec{x})}{\partial x_i}$ .

**Fact 2.11** ([6]) *Let  $G$  be an  $r$ -graph on  $[n]$ . Let  $\vec{x} = (x_1, x_2, \dots, x_n)$  be an optimum weight vector on  $G$ . Then*

$$\frac{\partial \lambda(G, \vec{x})}{\partial x_i} = r \lambda(G)$$

for every  $i \in [n]$  with  $x_i > 0$ .

**Fact 2.12** *Let  $G$  be an  $r$ -graph on  $[n]$ . Let  $\vec{x} = (x_1, x_2, \dots, x_n)$  be a feasible weight vector on  $G$ . Let  $i, j \in [n]$ , where  $i \neq j$ . Suppose that  $L_G(i \setminus j) = L_G(j \setminus i) = \emptyset$ . Let  $\vec{y} = (y_1, y_2, \dots, y_n)$  be defined by letting  $y_\ell = x_\ell$  for every  $\ell \in [n] \setminus \{i, j\}$  and letting  $y_i = y_j = \frac{1}{2}(x_i + x_j)$ . Then  $\lambda(G, \vec{y}) \geq \lambda(G, \vec{x})$ . Furthermore, if the pair  $\{i, j\}$  is not covered by any edge of  $G$  and  $\lambda(G, \vec{y}) = \lambda(G, \vec{x})$ , then  $x_i = x_j$ .*

*Proof.* Since  $L_G(i \setminus j) = L_G(j \setminus i) = \emptyset$ , we have

$$\lambda(G, \vec{y}) - \lambda(G, \vec{x}) = \sum_{\{i, j\} \subseteq e \in G} \left[ \frac{(x_i + x_j)^2}{4} - x_i x_j \right] \prod_{k \in e \setminus \{i, j\}} x_k \geq 0.$$

If the pair  $\{i, j\}$  is not covered by any edge of  $G$  then equality holds only if  $x_i = x_j$ . ■

As usual, if  $V_1, \dots, V_s$  are disjoint sets of vertices then  $\Pi_{i=1}^s V_i = V_1 \times V_2 \times \dots \times V_s = \{(x_1, x_2, \dots, x_s) : \forall i = 1, \dots, s, x_i \in V_i\}$ . We will use  $\Pi_{i=1}^s V_i$  to also denote the set of the corresponding unordered  $s$ -sets. If  $L$  is a hypergraph on  $[m]$ , then a *blowup* of  $L$  is a hypergraph  $G$  whose vertex set can be partitioned into  $V_1, \dots, V_m$  such that  $E(G) = \bigcup_{e \in L} \prod_{i \in e} V_i$ . The following proposition follows immediately from the definition and is implicit in many papers (see [11] for instance).

**Proposition 2.13** *Let  $r \geq 2$ . Let  $L$  be an  $r$ -graph and  $G$  a blowup of  $L$ . Suppose  $|V(G)| = n$ . Then  $|G| \leq \lambda(L)n^r$ .* ■

### 3 Lagrangian of an $r$ -graph not containing a $t$ -matching and related Turán numbers

#### 3.1 Lagrangian density of $M_t^3$

**Lemma 3.1** *Let  $n, r, t$  be positive integers where  $t \geq 2$  and  $n \geq r \geq 2$ . Let  $\mathcal{F}$  denote the family of all  $r$ -graphs  $H$  with no isolated vertex on at most  $n$  vertices such that  $H$  is  $M_t^T$ -free and  $H \neq K_{tr-1}^T$ . Then there*

exists a dense  $r$ -graph  $G \in \mathcal{F}$  and an optimum vector  $\vec{x}$  on  $G$  such that  $\lambda(G, \vec{x}) = \max\{\lambda(H) : H \in \mathcal{F}\}$  and that  $G$  is  $\vec{x}$ -compressed.

*Proof.* First note that if  $H \in \mathcal{F}$  then  $H$  is  $K_{tr-1}^r$ -free. Otherwise suppose  $H$  contains a copy  $K$  of  $K_{tr-1}^r$ . Then since  $H$  has no isolated vertex and  $H \neq K_{tr-1}^r$ ,  $H$  contains some edge not in  $K$ , in which case we can find a  $t$ -matching in  $H$ , a contradiction. Let  $\lambda^* = \max\{\lambda(H) : H \in \mathcal{F}\}$ . Let  $G_1 \in \mathcal{F}$  be an  $r$ -graph with  $\lambda(G_1) = \lambda^*$ . By Lemma 2.9 (b), there exists a  $M_t^r$ -free dense  $r$ -graph  $G'_1$  with  $|V(G'_1)| \leq |V(G_1)|$  such that  $\lambda(G'_1) \geq \lambda(G_1)$  and  $G'_1$  is  $\vec{x}$ -compressed, where  $\vec{x}$  is an optimum vector of  $G'_1$ . Furthermore,  $G'_1$  is  $K_{tr-1}^r$ -free. Hence  $G'_1 \in \mathcal{F}$ . So  $\lambda(G'_1) = \lambda^*$ . The claim thus holds by letting  $G = G'_1$ . ■

Hefetz and Keevash [7] established the Lagrangian density of  $M_2^3$ . We give a short new proof here.

**Theorem 3.2** ([7]) *Let  $G$  be an  $M_2^3$ -free 3-graph. Then  $\lambda(G) \leq \lambda(K_5^3) = \frac{2}{25}$ . Furthermore, if  $G \neq K_5^3$  and  $G$  has no isolated vertex, then  $\lambda(G) \leq \lambda(K_5^3) - 10^{-3}$ .*

*Proof.* (new proof) It suffices to prove that if  $G$  is an  $M_2^3$ -free 3-graph with no isolated vertex and  $G \neq K_5^3$  then  $\lambda(G) \leq \lambda(K_5^3) - 10^{-3}$ . By Lemma 3.1, it suffices to assume that  $G$  is dense and has an optimum weight vector  $\vec{x}$  such that  $G$  is  $\vec{x}$ -compressed. Suppose  $V(G) = [n]$ . If  $n \leq 5$ , then  $\lambda(G) \leq \lambda(K_5^{3-}) < \lambda(K_5^3) - 10^{-3}$ , where  $K_5^{3-}$  is the 3-graph obtained by removing one edge from  $K_5^3$ . Hence, we may assume that  $n \geq 6$ . By our assumption, there exists a linear order  $\mu$  on  $[n]$  such that  $\forall i, j \in [n]$  whenever  $i <_\mu j$  we have  $x_i \geq x_j$  and that  $G$  is compressed relative to  $\mu$ . By relabelling if needed, we may assume that  $\mu$  is the natural order  $1 < 2 < \dots < n$ . Then  $x_1 \geq x_2 \geq \dots \geq x_n$ . By Fact 2.5,  $G$  covers pairs. So  $i(n-1)n \in G$ , for some  $i < n-1$ . Since  $G$  is compressed relative to the natural order, we have  $1(n-1)n \in G$ . Again, since  $G$  is compressed relative to the natural order, this implies that  $\forall i, j$ , where  $2 \leq i < j \leq n$ ,  $1ij \in G$ . Suppose that  $G[\{2, \dots, n\}]$  contains an edge  $e$ . Since  $n \geq 6$ ,  $\exists i, j \in \{2, \dots, n\}$ , such that  $i, j \notin e$ . Now,  $\{1ij, e\}$  forms a 2-matching in  $G$ , contradicting  $G$  being  $M_2^3$ -free. Hence  $G = \{1ij : 2 \leq i < j \leq n\}$ . Assume that  $x_1 = a$ . Since  $\vec{y} = (\frac{x_2}{1-a}, \dots, \frac{x_n}{1-a})$  is a feasible weight vector on  $L_G(1)$ , by Theorem 2.10

$$\lambda(G) = \lambda(G, \vec{x}) = a(1-a)^2 \lambda(L_G(1), \vec{y}) < \frac{1}{2} a(1-a)^2 \leq \frac{1}{4} \left[ \frac{2a + (1-a) + (1-a)}{3} \right]^3 = \frac{2}{27} < \lambda(K_5^3) - 10^{-3}.$$

■

We now extend Theorem 3.2 to determine (with stability) the maximum Lagrangian of a 3-graph not containing a  $t$ -matching, for all  $t \geq 2$ . Given an  $r$ -graph  $G = (V, E)$  and  $i \in V$ , let

$$I_G(i) = \{e \in E : i \in e\}.$$

**Theorem 3.3** *Let  $t \geq 2$  be a positive integer. Let  $G$  be an  $M_t^3$ -free 3-graph with no isolated vertex and  $G \neq K_{3t-1}^3$ . Then there exists a positive real  $c_1 = c_1(t)$  such that  $\lambda(G) \leq \lambda(K_{3t-1}^3) - c_1 = \frac{1}{6} \left( \frac{[3t-1]_3}{(3t-1)^3} - 6c_1 \right)$ .*

*Proof.* By Lemma 3.1, it suffices to assume that  $G$  is dense and has an optimum weight vector  $\vec{x}$  such that  $G$  is  $\vec{x}$ -compressed. Suppose  $V(G) = [n]$ . Let  $K_{3t-1}^{3-}$  be the 3-graph obtained by removing one edge from  $K_{3t-1}^3$ . If  $n \leq 3t - 1$ , then since  $G \neq K_{3t-1}$ ,  $\lambda(G) \leq \lambda(K_{3t-1}^{3-})$ . So, we may assume that  $n \geq 3t$ . We use induction on  $t$ , with Theorem 3.2 forming the basis step  $t = 2$ . For the induction step, let  $t \geq 3$ . By our assumption, there exists a linear order  $\mu$  on  $[n]$  such that  $\forall i, j \in [n]$ ,  $x_i \geq x_j$  if  $i <_\mu j$  and that  $G$  is compressed relative to  $\mu$ . By relabelling if needed, we may assume that  $\mu$  is the natural order  $1 < 2 < \dots < n$ . Then  $x_1 \geq x_2 \geq \dots \geq x_n$ . By Fact 2.5,  $G$  covers pairs. So  $i(n-1)n \in G$ , for some  $i < n-1$ . Since  $G$  is compressed relative to the natural order, this implies  $1(n-1)n \in G$  and furthermore

$$I_G(1) = \{1ij : 2 \leq i < j \leq n\}. \quad (1)$$

Suppose  $x_1 = a$ . Then  $0 < a < 1$ . Since  $\vec{z} = (\frac{x_2}{1-a}, \dots, \frac{x_n}{1-a})$  is a feasible weight vector on  $L_G(1) = K_{n-1}^2$ . By Theorem 2.10, we have

$$\lambda(I_G(1), \vec{x}) = a \cdot \sum_{2 \leq i < j \leq n} x_i x_j = a(1-a)^2 \lambda(L_G(1), \vec{z}) < \frac{1}{2} a(1-a)^2.$$

Let  $F = G[\{2, 3, \dots, n\}]$ . Suppose  $F$  contains a  $(t-1)$ -matching  $M$ . Since  $n \geq 3t$ , there exist distinct vertices  $i, j \in [n] \setminus (V(M) \cup \{1\})$ . By (1),  $1ij \in G$ . Now,  $M \cup \{1ij\}$  is a  $t$ -matching in  $G$ , contradicting  $G$  being  $M_t^3$ -free. Hence  $F$  must be  $M_{t-1}^3$ -free. Note that  $\vec{z}$  is a feasible weight vector on  $F$ . By the induction hypothesis (by considering  $F = K_{3t-4}^3$  or not), we have  $\lambda(F, \vec{z}) \leq \lambda(K_{3t-4}^3)$ . Thus,

$$\lambda(F, \vec{x}) = (1-a)^3 \cdot \lambda(F, \vec{z}) \leq (1-a)^3 \lambda(F) \leq (1-a)^3 \lambda(K_{3t-4}^3) = \binom{3t-4}{3} \left( \frac{1-a}{3t-4} \right)^3.$$

Let  $s = 3t - 4$  and  $\mu = \frac{s^2 - 3s + 2}{6s^2}$ . We have

$$\begin{aligned} \lambda(G) = \lambda(G, \vec{x}) &\leq \lambda(I_G(1), \vec{x}) + \lambda(F, \vec{x}) \\ &< \frac{1}{2} a(1-a)^2 + \binom{s}{3} \left( \frac{1-a}{s} \right)^3 \\ &= \frac{1}{2} a(1-a)^2 + \frac{s^2 - 3s + 2}{6s^2} (1-a)^3 \\ &= (1-a)^2 \left( \frac{1}{2} a + \mu(1-a) \right) \\ &= (1-a)^2 \left( \left( \frac{1}{2} - \mu \right) a + \mu \right) \\ &= (1-a)(1-a) \left( 2a + \frac{\mu}{\frac{1}{4} - \frac{1}{2}\mu} \right) \cdot \left( \frac{1}{4} - \frac{1}{2}\mu \right) \\ &\leq \left[ \frac{1}{3} \left( 1-a + 1-a + 2a + \frac{\mu}{\frac{1}{4} - \frac{1}{2}\mu} \right) \right]^3 \cdot \left( \frac{1}{4} - \frac{1}{2}\mu \right) \quad (\text{by the AM-GM inequality}) \\ &= \frac{1}{54 \left( \frac{1}{2} - \mu \right)^2} \\ &= \frac{2s^4}{3(2s^2 + 3s - 2)^2}. \end{aligned}$$

Since  $s = 3t - 4$ , we have

$$\lambda(K_{3t-1}^3) = \binom{3t-1}{3} \left( \frac{1}{3t-1} \right)^3 = \binom{s+3}{3} \cdot \left( \frac{1}{s+3} \right)^3 = \frac{s^2 + 3s + 2}{6(s+3)^2}.$$



Hence,

$$\begin{aligned}
\lambda(G) - \lambda(K_{3t-1}^3) &\leq \frac{2s^4}{3(2s^2 + 3s - 2)^2} - \frac{s^2 + 3s + 2}{6(s + 3)^2} \\
&= \frac{4s^4(s + 3)^2 - (2s^2 + 3s - 2)^2(s^2 + 3s + 2)}{6(2s^2 + 3s - 2)^2(s + 3)^2} \\
&= -\frac{9s^4 + 15s^3 - 30s^2 - 12s + 8}{6(2s^2 + 3s - 2)^2(s + 3)^2},
\end{aligned}$$

which is negative for every  $s \geq 2$ . Let

$$c_1 = \min \left\{ \lambda(K_{3t-1}^3) - \lambda(K_{3t-1}^{3-}), \frac{9s^4 + 15s^3 - 30s^2 - 12s + 8}{6(2s^2 + 3s - 2)^2(s + 3)^2} \right\}.$$

Then  $\lambda(G) \leq \lambda(K_{3t+2}^3) - c_1$  and the proof is complete.  $\blacksquare$

**Corollary 3.4**  $\pi_\lambda(M_t^3) = 3!\lambda(K_{3t-1}^3) = \frac{[3t-1]_3}{(3t-1)^3}.$

*Proof.* Since  $K_{3t-1}^3$  is  $M_t^3$ -free,  $\pi_\lambda(M_t^3) \geq 3!\lambda(K_{3t-1}^3)$ . On the other hand, by Theorem 3.3,  $\pi_\lambda(M_t^3) \leq 3!\lambda(K_{3t-1}^3)$ . Therefore,  $\pi_\lambda(M_t^3) = 3!\lambda(K_{3t-1}^3)$ .  $\blacksquare$

### 3.2 Turán number of the extension of $M_t^3$

The main result in this section is as follows.

**Theorem 3.5** *Let  $t \geq 2$  be an integer. Then  $ex(n, H_{3t}^{M_t^3}) = t_{3t-1}^3(n)$  for sufficiently large  $n$ . Moreover, if  $n$  is sufficiently large and  $G$  is an  $H_{3t}^{M_t^3}$ -free 3-graph on  $[n]$  with  $|G| = t_{3t-1}^3(n)$ , then  $G = T_{3t-1}^3(n)$ .*

To prove the theorem, we need several results from [2]. Similar results are obtained independently in [15].

**Definition 3.6** ([2]) Let  $m, r \geq 2$  be positive integers. Let  $F$  be an  $r$ -graph that has at most  $m + 1$  vertices satisfying  $\pi_\lambda(F) \leq \frac{[m]_r}{m^r}$ . We say that  $\mathcal{K}_{m+1}^F$  is  $m$ -stable if for every real  $\varepsilon > 0$  there are a real  $\delta > 0$  and an integer  $n_1$  such that if  $G$  is a  $\mathcal{K}_{m+1}^F$ -free  $r$ -graph with at least  $n \geq n_1$  vertices and more than  $(\frac{[m]_r}{m^r} - \delta)\binom{n}{r}$  edges, then  $G$  can be made  $m$ -partite by deleting at most  $\varepsilon n$  vertices.

**Theorem 3.7** ([2]) *Let  $m, r \geq 2$  be positive integers. Let  $F$  be an  $r$ -graph that either has at most  $m$  vertices or has  $m + 1$  vertices one of which has degree 1. Suppose either  $\pi_\lambda(F) < \frac{[m]_r}{m^r}$  or  $\pi_\lambda(F) = \frac{[m]_r}{m^r}$  and  $\mathcal{K}_{m+1}^F$  is  $m$ -stable. Then there exists a positive integer  $n_2$  such that for all  $n \geq n_2$  we have  $ex(n, H_{m+1}^F) = t_m^r(n)$  and the unique extremal  $r$ -graph is  $T_m^r(n)$ .*  $\blacksquare$

Given an  $r$ -graph  $G$  and a real  $\alpha$  with  $0 < \alpha \leq 1$ , we say that  $G$  is  $\alpha$ -dense if  $G$  has minimum degree at least  $\alpha\binom{|V(G)|-1}{r-1}$ . Let  $i, j \in V(G)$ , we say  $i$  and  $j$  are *nonadjacent* if  $\{i, j\}$  is not contained in any edge of  $G$ . Given a set  $U \subseteq V(G)$ , we say  $U$  is an *equivalence class* of  $G$  if for every two vertices  $u, v \in U$ ,  $L_G(u) = L_G(v)$ . Given two nonadjacent nonequivalent vertices  $u, v \in V(G)$ , *symmetrizing  $u$  to  $v$*  refers to the operation of deleting all edges containing  $u$  of  $G$  and adding all the edges  $\{u\} \cup A, A \in L_G(v)$  to  $G$ . We use the following algorithm from [2], which was originated in [16].

**Algorithm 3.8** (Symmetrization and cleaning with threshold  $\alpha$ )

**Input:** An  $r$ -graph  $G$ .

**Output:** An  $r$ -graph  $G^*$ .

**Initiation:** Let  $G_0 = H_0 = G$ . Set  $i = 0$ .

**Iteration:** For each vertex  $u$  in  $H_i$ , let  $A_i(u)$  denote the equivalence class that  $u$  is in. If either  $H_i$  is empty or  $H_i$  contains no two nonadjacent nonequivalent vertices, then let  $G^* = H_i$  and terminate. Otherwise let  $u, v$  be two nonadjacent nonequivalent vertices in  $H_i$ , where  $d_{H_i}(u) \geq d_{H_i}(v)$ . We symmetrize each vertex in  $A_i(v)$  to  $u$ . Let  $G_{i+1}$  denote the resulting graph. If  $G_{i+1}$  is  $\alpha$ -dense, then let  $H_{i+1} = G_{i+1}$ . Otherwise we let  $L = G_{i+1}$  and repeat the following: let  $z$  be any vertex of minimum degree in  $L$ . We redefine  $L = L - z$  unless in forming  $G_{i+1}$  from  $H_i$  we symmetrized the equivalence class of some vertex  $v$  in  $H_i$  to some vertex in the equivalence class of  $z$  in  $H_i$ . In that case, we redefine  $L = L - v$  instead. We repeat the process until  $L$  becomes either  $\alpha$ -dense or empty. Let  $H_{i+1} = L$ . We call the process of forming  $H_{i+1}$  from  $G_{i+1}$  “cleaning”. Let  $Z_{i+1}$  denote the set of vertices removed, so that  $H_{i+1} = G_{i+1} - Z_{i+1}$ . By our definition, if  $H_{i+1}$  is nonempty then it is  $\alpha$ -dense.

**Theorem 3.9** ([2]) *Let  $m, r \geq 2$  be positive integers. Let  $F$  be an  $r$ -graph that has at most  $m$  vertices or has  $m + 1$  vertices one of which has degree 1. There exists a real  $\gamma_0 = \gamma_0(m, r) > 0$  such that for every positive real  $\gamma < \gamma_0$ , there exist a real  $\delta > 0$  and an integer  $n_0$  such that the following is true for all  $n \geq n_0$ . Let  $G$  be an  $\mathcal{K}_{m+1}^F$ -free  $r$ -graph on  $[n]$  with more than  $(\frac{[m]_r}{m^r} - \delta) \binom{n}{r}$  edges. Let  $G^*$  be the final  $r$ -graph produced by Algorithm 3.8 with threshold  $\frac{[m]_r}{m^r} - \gamma$ . Then  $|V(G^*)| \geq (1 - \gamma)n$  and  $G^*$  is  $(\frac{[m]_r}{m^r} - \gamma)$ -dense. Furthermore, if there is a set  $W \subseteq V(G^*)$  with  $|W| \geq (1 - \gamma_0)|V(G^*)|$  such that  $W$  is the union of a collection of at most  $m$  equivalence classes of  $G^*$ , then  $G[W]$  is  $m$ -partite. ■*

The following corollary is implicit in [2] and [15].

**Corollary 3.10** *Let  $m, r \geq 2$  be positive integers. Let  $F$  be an  $r$ -graph that has at most  $m + 1$  vertices with a vertex of degree 1 and  $\pi_\lambda(F) \leq \frac{[m]_r}{m^r}$ . Suppose there is a constant  $c > 0$  such that for every  $F$ -free  $r$ -graph  $L$  with no isolated vertex and  $L \neq K_m^r$ ,  $\lambda(L) \leq \lambda(K_m^r) - c$ . Then  $\mathcal{K}_{m+1}^F$  is  $m$ -stable.*

*Proof.* Let  $\varepsilon > 0$  be given. Let  $\delta, n_0$  be the constants guaranteed by Theorem 3.9. We can assume that  $\delta$  is small enough and  $n_0$  is large enough. Let  $\gamma > 0$  satisfy  $\gamma < \varepsilon$  and  $\delta + r\gamma < c$ . Let  $G$  be a  $\mathcal{K}_{m+1}^F$ -free  $r$ -graph on  $n > n_0$  vertices with more than  $(\frac{[m]_r}{m^r} - \delta) \binom{n}{r}$  edges. Let  $G^*$  be the final  $r$ -graph produced by applying Algorithm 3.8 to  $G$  with threshold  $\frac{[m]_r}{m^r} - \gamma$ . By Algorithm 3.8, if  $S$  consists of one vertex from each equivalence class of  $G^*$ , then  $G^*[S]$  covers pairs and  $G^*$  is a blowup of  $G^*[S]$ .

First, suppose that  $|S| \geq m + 1$ . If  $F \subseteq G^*[S]$ , then since  $G^*[S]$  covers pairs we can find a member of  $\mathcal{K}_{m+1}^F$  in  $G^*[S]$  by using any  $(m + 1)$ -set that contains a copy of  $F$  as the core, contradicting  $G^*$  being  $\mathcal{K}_{m+1}^F$ -free. So  $G^*[S]$  is  $F$ -free. Since  $|S| \geq m + 1$  and  $G^*[S]$  covers pairs, clearly  $G^*[S] \neq K_m^r$ . Also,  $G^*[S]$  has no isolated vertex. Hence, by our assumption,  $\lambda(G^*[S]) \leq \frac{1}{r!} \frac{[m]_r}{m^r} - c$ . By Proposition 2.13, we have

$$|G^*| \leq \lambda(G^*[S])n^r \leq \left(\frac{1}{r!} \frac{[m]_r}{m^r} - c\right)n^r < \left(\frac{[m]_r}{m^r} - c\right) \frac{n^r}{r!}. \quad (2)$$

Now, during the process of obtaining  $G^*$  from  $G$ , symmetrization never decreases the number of edges. Since at most  $\gamma n$  vertices are deleted in the process (see Theorem 3.9),

$$|G^*| > |G| - \gamma n \binom{n-1}{r-1} \geq \left(\frac{[m]_r}{m^r} - \delta - r\gamma\right) \binom{n}{r} > \left(\frac{[m]_r}{m^r} - c\right) \frac{n^r}{r!},$$

contradicting (2). So  $|S| \leq m$ . Hence,  $W = V(G^*)$  is the union of at most  $m$  equivalence classes of  $G^*$ . By Theorem 3.9,  $|W| \geq (1 - \gamma)n$  and  $G[W]$  is  $m$ -partite. Hence,  $G$  can be made  $m$ -partite by deleting at most  $\gamma n < \varepsilon n$  vertices. Thus,  $\mathcal{K}_{m+1}^F$  is  $m$ -stable. ■

**Proof of Theorem 3.5.** By Theorem 3.3 and Corollary 3.4,  $M_t^3$  satisfies the conditions of Corollary 3.10. So,  $\mathcal{K}_{3t}^{M_t^3}$  is  $(3t - 1)$ -stable. The theorem then follows from Theorem 3.7. ■

## 4 Local Lagrangians of $M_t^r$ -free $r$ -graphs and Lagrangians of $L_t^r$ -free $r$ -graphs and related Turán numbers

In this section we consider a local version of Lagrangians of  $M_t^r$ -free graphs for  $r = 2, 3$ . This will then be used to determine the Lagrangian density of a linear star  $L_t^r$  for  $r = 3, 4$ . Let  $0 < b < 1$  be a real. Given an  $r$ -graph  $G$  on  $[n]$ , a feasible weight vector  $\vec{x} = (x_1, \dots, x_n)$  is called a  *$b$ -bounded feasible weight vector* on  $G$  if  $\forall i \in [n]$ ,  $x_i \leq b$ . If  $G$  has a  $b$ -bounded feasible weight vector, then we define the  *$b$ -bounded Lagrangian* of  $G$  as

$$\lambda_b(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \text{ is a } b\text{-bounded feasible weight vector on } G\}. \quad (3)$$

If  $G$  does not have any  $b$ -bounded feasible weight vector, then we define  $\lambda_b(G) = 0$ . A feasible  $b$ -bounded weight vector  $\vec{x}$  on  $G$  such that  $\lambda(G, \vec{x}) = \lambda_b(G)$  is called an *optimum  $b$ -bounded weight vector* on  $G$ . We now consider  $\lambda_b(G)$  over  $M_t^r$ -free  $r$ -graphs for  $r = 2, 3$  for appropriate values of  $b$ . For such a study, first we reduce the problem to the case where the  $r$ -graph in consideration is compressed and there exists an optimum  $b$ -bounded weight vector with some additional properties.

**Lemma 4.1** *Let  $0 < b < 1$  be a real. Let  $r, t \geq 2$  be integers. Let  $\mathcal{F}$  be the family of all  $M_t^r$ -free  $r$ -graphs. There exists  $G \in \mathcal{F}$  and an optimum  $b$ -bounded weight vector  $\vec{x}$  on  $G$  such that*

1.  $\lambda(G, \vec{x}) = \lambda_b(G) = \max\{\lambda_b(H) : H \in \mathcal{F}\}$ .
2.  $G$  is  $\vec{x}$ -compressed.
3. All vertices of  $G$  have positive weight under  $\vec{x}$ .
4. If  $u, v$  are any two vertices in  $G$  with weight less than  $b$  under  $\vec{x}$  then  $\{u, v\}$  is covered in  $G$ .

*Proof.* Clearly,  $\mathcal{F}$  is closed under taking subgraphs. Let  $\lambda^* = \max\{\lambda_b(H) : H \in \mathcal{F}\}$ . Among all  $r$ -graphs  $H \in \mathcal{F}$  with  $\lambda_b(H) = \lambda^*$ , let  $G$  be the one with the fewest possible vertices. Let  $\vec{x}$  be an optimum  $b$ -bounded weight vector on  $G$  that has the maximum number of  $b$ -components. By Lemma 2.9 (a), we may assume that  $G$  is  $\vec{x}$ -compressed (or else we could replace  $G$  with one that is  $\vec{x}$ -compressed). If some vertex in  $G$  has 0 weight under  $\vec{x}$  then deleting that vertex would give us a graph  $G' \in \mathcal{F}$  with  $\lambda_b(G') = \lambda^*$  and having fewer vertices than  $G$ , contradicting our choice of  $G$ . Hence, all vertices in  $G$  have positive weights under  $\vec{x}$ . Now, suppose  $u, v$  are two vertices with weight less than  $b$  under  $\vec{x}$ . Suppose that no edge of  $G$  contains both  $u$  and  $v$ . Without loss of generality suppose that  $\lambda(L_G(u), \vec{x}) \geq \lambda(L_G(v), \vec{x})$ . If we decrease the weight of  $v$  and increase the weight of  $u$  by the same amount, the total weight does not decrease. Hence, we can obtain an optimum  $b$ -bounded weight vector on  $G$  that either has more

$b$ -components than  $\vec{x}$  or has weight 0 on  $v$ . In the former, we get a contradiction to our choice of  $\vec{x}$ . In the latter case, we get a contradiction to our choice of  $G$ . Hence there must be some edge in  $G$  containing both  $u$  and  $v$ . ■

For the purpose of studying  $L_t^r$ -free graphs, we will also need the following short lemma.

**Lemma 4.2** *Let  $r, t \geq 2$ . Let  $G$  be an  $L_t^r$ -free  $r$ -graph with at least  $t(r-1) + 1$  vertices and  $G$  covers pairs. Let  $x \in V(G)$ . Then  $L(x)$  is  $K_{t(r-1)-1}^{r-1}$ -free. In particular,  $G$  is  $K_{t(r-1)}^r$ -free.*

*Proof.* Suppose for contradiction that  $L(x)$  contains a copy  $K$  of  $K_{t(r-1)-1}^{r-1}$ . By our assumption,  $\exists y \in V(G) \setminus (V(K) \cup \{x\})$ . Since  $G$  covers pairs, there exists  $e \in G$  that contains  $x$  and  $y$ . Now we can find a copy of  $L_{t-1}^r$  using a  $(t-1)$ -matching in  $K$  containing  $x$  that are disjoint from  $e \setminus \{x, y\}$ , which together with  $e$  form a copy of  $L_t^r$  in  $G$ , a contradiction. ■

#### 4.1 Local Lagrangians of $M_t^2$ -free graphs and Lagrangians of $L_t^3$ -free 3-graphs and related Turán numbers

We start the subsection by developing some structural properties of  $M_t^2$ -free left-compressed graphs. Let  $n, t$  be positive integers, where  $t \geq 2$  and  $n \geq 2t$ . For each  $\ell \in [t-1] \cup \{0\}$ , define

$$F_{t,\ell}(n) = \binom{[2t-1-\ell]}{2} \cup \{ab : a \in \{1, \dots, \ell\}, b \in \{2t-\ell, \dots, n\}\}.$$

Note that  $F_{t,\ell}(n)$  is  $M_t^2$ -free for each  $\ell \in [t-1] \cup \{0\}$ .

**Lemma 4.3** *Let  $n, t$  be positive integers, where  $t \geq 2$  and  $n \geq 2t$ . Let  $G$  be an  $M_t^2$ -free 2-graph on  $[n]$  that is left-compressed relative to the natural order. Then  $G \subseteq F_{t,\ell}(n)$  for some  $\ell \in [t-1] \cup \{0\}$ .*

*Proof.* For each  $i \in [t]$ , let  $N_i = \{j \in [n] : j > i, ij \in G\}$ . Since  $G$  is left-compressed relative to the natural order on  $[n]$ , we have either  $N_i = \emptyset$  or  $N_i = \{i+1, i+2, \dots, m_i\}$  for some  $m_i > i$ . Furthermore,  $N_1 \supseteq N_2 \supseteq \dots \supseteq N_t$ . For convenience, we define  $m_i = 1$  for those  $i \in [t]$  with  $N_i = \emptyset$ . Then  $\{m_1, \dots, m_t\}$  is non-increasing. Let  $h$  be the largest  $i \in [t]$  such that  $m_i \leq 2t-i$ . Note that  $h$  exists; otherwise  $\{i(2t+1-i) : i \in [t]\}$  is a  $t$ -matching in  $G$ , a contradiction. Let  $\ell = h-1$ . Then  $\ell \in [t-1] \cup \{0\}$ . By our assumption, there is no edge from  $[\ell+1, n]$  to  $[2t-\ell, n]$ . So  $G \subseteq F_{t,\ell}(n)$ . ■

**Lemma 4.4** *Let  $n, t$  be positive integers, where  $t \geq 2$  and  $n \geq 2t$ . Let  $b$  be a real such that  $0 < b \leq \frac{1}{t}$ . For each  $\ell \in [t-1]$ , we have  $\lambda_b(F_{t,\ell}(n)) \leq \binom{2t-1-2\ell}{2} b^2 + \ell b - \frac{\ell^2+\ell}{2} b^2$ .*

*Proof.* Let  $\ell \in [t-1]$ . Let  $\vec{x} = (x_1, \dots, x_n)$  be a  $b$ -bounded feasible vector on  $F_{t,\ell}(n)$  such that  $\lambda(F_{t,\ell}(n), \vec{x}) = \lambda_b(F_{t,\ell}(n))$ . Using Fact 2.12 (note that any new weight vector produced by Fact 2.12 based on  $\vec{x}$  is also  $b$ -bounded), we may assume that  $x_1 = \dots = x_\ell, x_{\ell+1} = \dots = x_{2t-1-\ell}$  and  $x_{2t-\ell} =$

$\dots = x_n$ . Let  $a = x_1$ ,  $c = x_{\ell+1}$ , and  $d = x_{2t-\ell} + \dots + x_n = 1 - \ell a - (2t-1-2\ell)c$ . We have

$$\begin{aligned}
\lambda(F_{t,\ell}(n), \vec{x}) &= \binom{\ell}{2} a^2 + \binom{2t-1-2\ell}{2} c^2 + (2t-1-2\ell)\ell ac + \ell a[1 - \ell a - (2t-1-2\ell)c] \\
&= \binom{\ell}{2} a^2 + \binom{2t-1-2\ell}{2} c^2 + \ell a(1 - \ell a) \\
&= \binom{2t-1-2\ell}{2} c^2 + \ell a - \frac{\ell^2 + \ell}{2} a^2 \\
&\leq \binom{2t-1-2\ell}{2} b^2 + \ell b - \frac{\ell^2 + \ell}{2} b^2,
\end{aligned}$$

where we used the fact that  $f(x) = \ell x - \frac{\ell^2 + \ell}{2} x^2$  is increasing on  $(-\infty, \frac{1}{\ell+1})$  and that  $a, c \leq b \leq \frac{1}{\ell+1}$ .  $\blacksquare$

**Theorem 4.5** *Let  $t \geq 2$  be an integer. If  $G$  is an  $L_t^3$ -free 3-graph, then  $\lambda(G) \leq \lambda(K_{2t}^3)$ . Furthermore, there is  $c_2 = c_2(t) > 0$  such that if  $G$  is an  $L_t^3$ -free 3-graph that covers pairs and  $G \neq K_{2t}^3$  then  $\lambda(G) \leq \lambda(K_{2t}^3) - c_2 = \frac{(2t-1)(t-1)}{12t^2} - c_2$ .*

*Proof.* It suffices to assume that  $G$  is dense (otherwise we consider an appropriate subgraph). So  $G$  covers pairs. In this set up, it suffices to prove the second statement. So assume that  $G$  covers pairs and  $G \neq K_{2t}^3$ . Suppose  $V(G) = [n] \cup \{0\}$ . If  $n < 2t$ , then  $\lambda(G) \leq \lambda(K_{2t}^{3-}) \leq \lambda(K_{2t}^3) - c_2$ , by choosing  $c_2$  to be small enough, where  $K_{2t}^{3-}$  denotes  $K_{2t}^3$  minus an edge. Hence, we may assume that  $n \geq 2t$ . Let  $\vec{x} = (x_0, x_1, \dots, x_n)$  be an optimum weight vector on  $G$ . Let  $a = \max\{x_i : i \in V(G)\}$ . By relabeling if needed, we may assume that  $x_0 = a$ . By Fact 2.11,  $\lambda(L(0), \vec{x}) = \frac{\partial \lambda(G, \vec{x})}{\partial x_0} = 3\lambda(G)$ , so it suffices to show that  $\lambda(L(0), \vec{x}) \leq \frac{(2t-1)(t-1)}{4t^2} - 3c_2$ , for some sufficiently small positive real  $c_2$ .

Since  $G$  is  $L_t^3$ -free,  $L(0)$  is  $M_t^2$ -free. Since  $G$  covers pairs and  $n \geq 2t$ , by Lemma 4.2,  $K_{2t-1}^2 \not\subseteq L(0)$ . We may view  $L(0)$  as a 2-graph on  $[n]$ . Let  $\vec{y} = (\frac{x_1}{1-a}, \dots, \frac{x_n}{1-a})$ . Then  $\vec{y}$  is a feasible weight vector on  $L(0)$ . Furthermore, it is  $\frac{a}{1-a}$ -bounded. We consider two cases.

**Case 1.**  $a \geq \frac{1}{2t}$ .

Since  $L(0)$  is  $K_{2t-1}^2$ -free, by Theorem 2.10,  $\lambda(L(0)) \leq \frac{1}{2}(1 - \frac{1}{2t-2})$ . Hence, for sufficiently small  $c_2 > 0$ ,

$$\lambda(L(0), \vec{x}) = (1-a)^2 \lambda(L(0), \vec{y}) \leq (1-a)^2 \lambda(L(0)) \leq \left(\frac{2t-1}{2t}\right)^2 \frac{1}{2} \frac{2t-3}{2t-2} < \frac{(2t-1)(t-1)}{4t^2} - 3c_2.$$

**Case 2.**  $a < \frac{1}{2t}$ .

Let  $b = \frac{a}{1-a}$ . Then  $b < \frac{1}{2t-1} \leq \frac{1}{t}$ . By Lemma 2.9 (a), there exists a  $M_t^2$ -free 2-graph  $H$  on  $[n]$  such that  $\lambda(H, \vec{y}) \geq \lambda(L(0), \vec{y})$  and such that  $H$  is  $\vec{y}$ -compressed. Also, since  $L(0)$  is  $K_{2t-1}^2$ -free, by Lemma 2.4,  $H$  is also  $K_{2t-1}^2$ -free. By relabeling if needed, we may assume that  $y_1 \geq \dots \geq y_n$  and that  $H$  is left-compressed relative to the natural order on  $[n]$ . By Lemma 4.3,  $H \subseteq F_{t,\ell}(n)$  for some  $\ell \in [t-1] \cup \{0\}$ . First, assume that  $\ell \in [t-1]$ .

Since  $\vec{y}$  is a  $b$ -bounded feasible weight vector on  $[n]$ , by Lemma 4.4, we have

$$\begin{aligned}
\lambda(L(0), \vec{x}) &= (1-a)^2 \lambda(L(0), \vec{y}) \leq \lambda(H, \vec{y}) \leq \lambda(F_{t,\ell}(n), \vec{y}) \\
&\leq (1-a)^2 \left[ \binom{2t-1-2\ell}{2} \left(\frac{a}{1-a}\right)^2 + \ell \frac{a}{1-a} - \frac{\ell^2 + \ell}{2} \left(\frac{a}{1-a}\right)^2 \right] \\
&= \binom{2t-1-2\ell}{2} a^2 + \ell a(1-a) - \frac{\ell^2 + \ell}{2} a^2.
\end{aligned}$$

Since  $f(x) = \ell x(1-x) - \frac{\ell^2 + \ell}{2} x^2$  increases on  $(-\infty, \frac{1}{\ell+3})$  and  $a < \frac{1}{2t} \leq \frac{1}{\ell+3}$ , we have

$$\begin{aligned}
\lambda(L(0), \vec{x}) &\leq \binom{2t-1-2\ell}{2} \left(\frac{1}{2t}\right)^2 + \ell \frac{1}{2t} \left(1 - \frac{1}{2t}\right) - \frac{\ell^2 + \ell}{2} \left(\frac{1}{2t}\right)^2 \\
&= \binom{2t-1-2\ell}{2} \left(\frac{1}{2t}\right)^2 + \ell \frac{1}{2t} \left(1 - \frac{2}{2t}\right) - \binom{\ell}{2} \left(\frac{1}{2t}\right)^2 \\
&= \lambda(F_{t,\ell}(2t-1), \vec{z}),
\end{aligned}$$

where  $z$  is a weight vector on  $[2t-1]$  with  $z = (\frac{1}{2t}, \dots, \frac{1}{2t})$ . Since  $\ell \geq 1$ ,  $F_{t,\ell}(2t-1) \subseteq K_{2t-1}^{2-}$ . Hence,

$$\lambda(L(0), \vec{x}) \leq \lambda(K_{2t-1}^{2-}, \vec{z}) \leq \frac{(2t-1)(t-1)}{4t^2} - 3c_2, \quad (4)$$

for sufficiently small  $c_2 > 0$ .

Finally, suppose  $\ell = 0$ . Note that  $F_{t,0}(n)$  consists of a copy of  $K_{2t-1}^2$  and some isolated vertices. Since  $H \subseteq F_{t,0}(n)$  and  $H$  is  $K_{2t-1}^2$ -free, we have  $\lambda(L(0), \vec{x}) \leq \lambda(H, \vec{x}) \leq \lambda(H) \leq \lambda(K_{2t-1}^{2-})$ . Hence (4) still holds for sufficiently small  $c_2 > 0$ . This completes our proof.  $\blacksquare$

**Corollary 4.6**  $\pi_\lambda(L_t^3) = 3! \lambda(K_{2t}^3) = \frac{[2t]_3}{(2t)^3}$ .

Applying Theorem 4.5, Corollary 4.6, Corollary 3.10 and Theorem 3.7, we have

**Theorem 4.7** *Let  $t \geq 2$  be an integer. Then  $ex(n, H_{2t+1}^{L_t^3}) = t_{2t}^3(n)$  for sufficiently large  $n$ . Moreover, if  $n$  is sufficiently large and  $G$  is an  $H_{2t+1}^{L_t^3}$ -free 3-graph on  $n$  vertices with  $|G| = t_{2t}^3(n)$  then  $G = T_{2t}^3(n)$ .*  $\blacksquare$

Theorem 4.7 is part of a more general theorem obtained in [2] and [15]. However, the method we used in this section is self-contained and is very different from those used in [2] and [15].

## 4.2 Local Lagrangians of $M_t^3$ -free 3-graphs and Lagrangians of $L_t^4$ -free 4-graphs and related Turán numbers

Next, we consider local Lagrangians of  $M_t^3$ -free 3-graphs. First, we focus on the  $t = 2$  case. As before, we first develop some structural properties of  $M_2^3$ -free 3-graphs. Given a 3-graph  $G$  on  $[n]$ , let  $L^+(1)$  and  $L^+(2)$  denote the links of 1, 2 of  $G$  in  $[3, n]$  respectively, i.e.

$$L^+(i) = \{A \subseteq [3, n] : A \cup \{i\} \in G\}$$

for  $i = 1, 2$ . We say a set  $S \subseteq V(G)$  is a *vertex cover* of  $G$  if for every edge  $e$  of  $G$ ,  $e \cap S \neq \emptyset$ .

**Lemma 4.8** *Let  $n \geq 6$  be an integer. Let  $G$  be an  $M_2^3$ -free 3-graph on  $[n]$  with no isolated vertex that is left-compressed relative to the natural order on  $[n]$ . Then*

- (a)  $\forall i \in [3, n], 12i \in G$ ,
- (b)  $\{1, 2\}$  is a vertex cover of  $G$ , and
- (c)  $L^+(2)$  is  $M_2^2$ -free. Thus, if  $L^+(2) \neq \emptyset$  then  $L^+(2)$  is either a triangle or a star.

*Proof.* By our assumption, for some  $i < j < n$ ,  $ijn \in G$ . Since  $G$  is left-compressed relative to the natural order on  $[n]$ , we have  $12n \in G$ . Since  $G$  is left-compressed, this further implies that  $12i \in G$  for every  $i \in [3, n]$ . If  $G$  contains an edge  $e$  not containing 1 or 2, then  $\{12i, e\}$  would form a 2-matching in  $G$ , for some  $i \in [n], i \notin e$  and  $i \neq 1, 2$ , contradicting  $G$  being  $M_2^3$ -free. Hence  $\{1, 2\}$  is a vertex cover of  $G$ . Finally, since  $G$  is left-compressed,  $L^+(2) \subseteq L^+(1)$ . If  $L^+(2)$  contains a 2-matching, then we would obtain a 2-matching in  $G$ , a contradiction. So  $L^+(2)$  is intersecting and must be either a star or a triangle. ■

Lemma 4.8 allows us to describe all left-compressed  $M_2^3$ -free 3-graphs on  $[n]$ .

**Definition 4.9** *For all integers  $n \geq 5$ , let*

$$\begin{aligned}
G_0(n) &= \{1ij : 2 \leq i < j \leq n\}, \\
G_1(n) &= \{12i : 3 \leq i \leq n\} \cup \{134, 135, 145, 234, 235, 245\}, \\
G_2(n) &= \binom{[4]}{3} \cup \{12i, 13i, 14i : 5 \leq i \leq n\}, \\
G_3(n) &= \{12i : 3 \leq i \leq n\} \cup \{13i : 4 \leq i \leq n\} \cup \{234, 235, 145\}, \\
G_4(n) &= \{12i : 3 \leq i \leq n\} \cup \{13i : 4 \leq i \leq n\} \cup \{23i : 4 \leq i \leq n\}.
\end{aligned}$$

**Lemma 4.10** *Let  $n \geq 6$  be an integer. Let  $G$  be an  $M_2^3$ -free 3-graph on  $[n]$  that is left-compressed relative to the natural order on  $[n]$ . Then  $G$  is a subgraph of one of  $G_0(n), G_1(n), G_2(n), G_3(n), G_4(n)$  given in Definition 4.9.*

*Proof.* By Lemma 4.8,  $\{1, 2\}$  is a vertex cover of  $G$  and  $L^+(2)$  is either empty, or a triangle or a star. We now consider three cases.

**Case 1.**  $L^+(2) = \emptyset$ .

Since  $G$  is left-compressed,  $L^+(i) = \emptyset$  for all  $i \geq 2$ . Hence  $G \subseteq G_0(n) = \{1ij : 2 \leq i < j \leq n\}$ .

**Case 2.**  $L^+(2)$  is a triangle.

Since  $G$  is left-compressed, we have  $L^+(2) = \{34, 35, 45\}$  and  $L^+(1) \supseteq L^+(2)$ . Since  $G$  contains no 2-matching, we must have  $L^+(1) = L^+(2) = \{34, 35, 45\}$ . Hence

$$G \subseteq G_1(n) = \{12i : 3 \leq i \leq n\} \cup \{134, 135, 145, 234, 235, 245\}.$$

**Case 3.**  $L^+(2)$  is a star.

Since  $G$  is left-compressed, we have  $L^+(2) = \{34, 35, \dots, 3p\}$  for some  $4 \leq p \leq n$ . Since  $G$  contains no 2-matching, every member of  $L(1 \setminus 2)$  must contain either 3 or 4. Further, if  $p \geq 6$  then every member of  $L(1 \setminus 2)$  must contain 3.

If  $p = 4$ , then

$$G \subseteq G_2(n) = \binom{[4]}{3} \cup \{12i, 13i, 14i : 5 \leq i \leq n\}.$$

If  $p = 5$ , then

$$G \subseteq G_3(n) = \{12i : 3 \leq i \leq n\} \cup \{13i : 4 \leq i \leq n\} \cup \{234, 235, 145\}.$$

If  $p \geq 6$ , then

$$G \subseteq G_4(n) = \{12i : 3 \leq i \leq n\} \cup \{13i : 4 \leq i \leq n\} \cup \{23i : 4 \leq i \leq n\}.$$

■

Let us recall the definition of the  $b$ -bounded Lagrangian  $\lambda_b(G)$  of  $G$ , given in (3).

**Lemma 4.11** *Let  $b$  be a real with  $0 < b \leq \frac{1}{3}$ . Let  $G$  be a 3-uniform star. Then  $\lambda_b(G) \leq \frac{1}{2}b(1-b)^2$ .*

*Proof.* Suppose  $V(G) = [n]$ . Without loss of generality suppose vertex 1 is the center of the star. Let  $\vec{x}$  be a  $b$ -bounded feasible vector on  $G$  with  $\lambda(G, \vec{x}) = \lambda_b(G)$ . Let  $a = x_1$ . Then  $a \leq b$ . Note that  $(\frac{x_2}{1-a}, \dots, \frac{x_n}{1-a})$  is a feasible weight vector on  $L_G(1)$ . By Theorem 2.10,  $\lambda(G, \vec{x}) \leq a \cdot \frac{1}{2}(1-a)^2 \leq \frac{1}{2}b(1-b)^2$ , where the last inequality follows from the fact that the function  $\frac{1}{2}x(1-x)^2$  increases on  $[0, \frac{1}{3}]$  and that  $0 < a \leq b \leq \frac{1}{3}$ . ■

**Lemma 4.12** *Let  $G$  be an  $M_2^3$ -free 3-graph. For  $0 < b \leq \frac{1}{5}$ , we have*

$$\lambda_b(G) \leq \max\{\frac{1}{2}b(1-b)^2, b^2 + 4b^3\}.$$

*Furthermore, if  $0 < b \leq \frac{1}{7}$  then  $\lambda_b(G) \leq \frac{1}{2}b(1-b)^2$ .*

*Proof.* Suppose  $V(G) = [n]$ . By Lemma 4.1, we may assume that  $G$  has an optimum  $b$ -bounded weight vector  $\vec{x}$  such that  $G$  is  $\vec{x}$ -compressed, all vertices of  $G$  have positive weights under  $\vec{x}$ , and such that all pairs of vertices of weight less than  $b$  are covered in  $G$ . By relabeling the vertices of  $G$  if needed we may assume that  $x_1 \geq \dots \geq x_n$  and that  $G$  is left-compressed relative to the natural order on  $[n]$ .

**Case 1.**  $x_{n-1} < b$ .

In this case we have  $x_{n-1}, x_n < b$ . By our assumption,  $\{n-1, n\}$  is covered in  $G$ . Since  $G$  is left-compressed, this implies that  $\forall 2 \leq i < j \leq n, 1ij \in G$ . If there is an edge of  $G'$  in  $\{2, \dots, n\}$  then since  $G$  is left-compressed, we have  $234 \in G$ . But then  $234, 156$  forms a  $M_2^3$  in  $G'$ , contradiction. Hence  $G' \subseteq G_0(n) = \{1ij : 2 \leq i < j \leq n\}$ . By Lemma 4.11,  $\lambda_b(G') \leq \lambda_b(G_0(n)) \leq \frac{1}{2}b(1-b)^2$ .

**Case 2.**  $x_{n-1} = b$ .

In this case we have  $x_1 = x_2 = \dots = x_{n-1} = b, x_n \leq b$ . By Lemma 4.10,  $G' \subseteq G_i$  for some  $i = 0, 1, 2, 3, 4$ . Since  $\lambda_b(G_0(n)) \leq \frac{1}{2}b(1-b)^2$ , we may assume that  $G' \subseteq G_i(n)$  for some  $i \in [4]$ . Since  $G_1(n) = \{12i : 3 \leq i \leq n\} \cup \{134, 135, 145, 234, 235, 245\}$ ,

$$\lambda(G_1(n), \vec{x}) \leq 6b^3 + b^2(1-2b) = b^2 + 4b^3.$$



Since  $G_2(n) = \binom{[4]}{3} \cup \{12i, 13i, 14i : 5 \leq i \leq n\}$ ,

$$\lambda(G_2(n), \vec{x}) \leq 4b^3 + 3b^2(1 - 4b) = 3b^2 - 8b^3.$$

Since  $G_3(n) = \{12i : 3 \leq i \leq n\} \cup \{13i : 4 \leq i \leq n\} \cup \{234, 235, 145\}$ ,

$$\lambda(G_3(n), \vec{x}) \leq b^2(1 - 2b) + b^2(1 - 3b) + 3b^3 = 2b^2 - 2b^3.$$

Since  $G_4(n) = \{12i : 3 \leq i \leq n\} \cup \{13i : 4 \leq i \leq n\} \cup \{23i : 4 \leq i \leq n\}$ ,

$$\lambda(G_4(n), \vec{x}) \leq b^3 + 3b^2(1 - 3b) = 3b^2 - 8b^3.$$

So

$$\lambda(G', \vec{x}) \leq \max\{\frac{1}{2}b(1 - b)^2, b^2 + 4b^3, 3b^2 - 8b^3, 2b^2 - 2b^3\} = \max\{\frac{1}{2}b(1 - b)^2, b^2 + 4b^3, 3b^2 - 8b^3\}.$$

Note that  $\frac{1}{2}b(1 - b)^2 - (3b^2 - 8b^3) \geq 0$  on  $[0, \infty)$ . Also,  $\frac{1}{2}b(1 - b)^2 - (b^2 + 4b^3) \geq 0$  on  $[0, \frac{1}{7}]$ . The conclusion follows.  $\blacksquare$

Next, we establish an upper bound on  $\lambda_b(G)$  for  $M_t^3$ -free graphs  $G$ , where  $t \geq 3$ . We need the following lemma of Frankl.

**Lemma 4.13** [3] *If  $G$  is an  $n$ -vertex  $r$ -graph with matching number  $s$  then  $|G| \leq s \binom{n-1}{r-1}$ .*  $\blacksquare$

**Lemma 4.14** *Let  $n, r, t$  be positive integers, where  $r, t \geq 2$ ,  $n \geq tr$ . Let  $G$  be an  $M_t^r$ -free graph on  $[n]$  that is left-compressed relative to the natural order. Then  $L_G(n)$  is  $M_t^{r-1}$ -free. Furthermore, if  $r = 3$  and  $\{n - 1, n\}$  is covered then  $G[\{2, \dots, n\}]$  is  $M_{t-1}^3$ -free.*

*Proof.* Suppose for contradiction that  $M = \{f_1, \dots, f_t\}$  is a  $t$ -matching in  $L_G(n)$ . Together they cover  $t(r - 1)$  vertices in  $[n - 1]$ . Since  $n \geq tr$ , there exist distinct vertices  $v_1, \dots, v_{t-1} \in [n - 1]$  that are not covered by  $M$ . Since  $G$  is left-compressed,  $f_1 \cup \{v_1\}, \dots, f_{t-1} \cup \{v_{t-1}\} \in G$ , which together with  $f_t \cup \{n\}$ , form a  $t$ -matching in  $G$ , a contradiction.

Next, suppose  $r = 3$  and  $\{n - 1, n\}$  is covered. Since  $G$  is left-compressed we have  $\forall 2 \leq i < j \leq n, 1ij \in G$ . Suppose  $G[\{2, \dots, n\}]$  contains  $(t - 1)$ -matching  $M$ . Then since  $n \geq 3t$ ,  $[n] \setminus \{1\}$  contains two vertices  $j, \ell$  not covered by  $M$ . Now,  $M \cup \{1j\ell\}$  is a  $t$ -matching in  $G$ , a contradiction.  $\blacksquare$

**Lemma 4.15** *Let  $t \geq 3$ . Let  $G$  be an  $M_t^3$ -free 3-graph. Let  $0 < b < \frac{1}{3t-1}$ . Let  $\vec{x}$  be a  $b$ -bounded feasible weight vector on  $G$  such that all but one of the components of  $\vec{x}$  are  $b$ . Then*

$$\lambda(G, \vec{x}) \leq \frac{t-1}{2}b(1 - 3b + 4b^2).$$

*Proof.* Suppose  $V(G) = [n]$ . Note that  $n \geq 3t$ . By Lemma 2.9 (a), we may assume that  $G$  is  $\vec{x}$ -compressed. By relabeling the vertices of  $G$  if needed we may assume that  $x_1 \geq \dots \geq x_n$  and that  $G$  is left-compressed relative to the natural order on  $[n]$ . By our assumption,  $x_1 = \dots = x_{n-1} = b$ . Suppose  $x_n = \alpha b$ , where  $0 < \alpha \leq 1$ . By Lemma 4.14,  $L(n)$  is  $M_t^2$ -free. Hence by Lemma 4.13,  $|L(n)| \leq (t - 1)(n - 1)$ . Let  $G'$  denote the set of edges of  $G$  not containing  $n$ . Since  $G'$  is  $M_t^3$ -free, by Lemma 4.13,  $|G'| \leq (t - 1)\binom{n-2}{2}$ . Hence the contribution to  $\lambda(G, \vec{x})$  of edges in  $G$  containing  $n$  or not

containing  $n$  are at most  $(t-1)(n-1)b^2 \cdot \alpha b$  and  $(t-1)\binom{n-2}{2}b^3$  respectively. Note that  $(n-1)b + \alpha b = 1$ . Also, on  $[0, 1]$  we have  $\alpha^2 - 3\alpha + \frac{1}{4} \geq -\frac{7}{4}$ . Hence

$$\begin{aligned}
\lambda(G, \vec{x}) &\leq (t-1)\binom{n-2}{2} \cdot b^3 + (t-1)\alpha(n-1)b^3 \\
&= \frac{t-1}{2}(n^2 - 5n + 6 + 2\alpha n - 2\alpha)b^3 \\
&= \frac{t-1}{2}\left((n - \frac{5}{2} + \alpha)^2 - (\frac{1}{4} - 3\alpha + \alpha^2)\right)b^3 \\
&= \frac{t-1}{2}\left((1 - \frac{3}{2}b)^2b - (\frac{1}{4} - 3\alpha + \alpha^2)b^3\right) \\
&\leq \frac{t-1}{2}\left((1 - \frac{3}{2}b)^2b + \frac{7}{4}b^3\right) \\
&= \frac{t-1}{2}b(1 - 3b + 4b^2).
\end{aligned}$$

■

**Lemma 4.16** *Let  $t \geq 3$  be an integer and  $b$  a real with  $0 < b < \frac{1}{3t-1}$ . Let  $G$  be an  $M_t^3$ -free 3-graph with  $n \geq 3t$  vertices. Then*

$$\lambda_b(G) \leq \frac{t-1}{2}b(1 - 3b + 6b^2).$$

*Proof.* Suppose  $V(G) = [n]$ . If no  $b$ -bounded feasible weight vector exists, then  $\lambda_b(G) = 0$  by definition and the claim holds trivially. So assume that there exist  $b$ -bounded feasible weight vectors. By Lemma 4.1, we may assume that  $G$  has an optimum  $b$ -bounded weight vector  $\vec{x}$  such that  $G$  is  $\vec{x}$ -compressed, all vertices of  $G$  have positive weights under  $\vec{x}$ , and such that all pairs of vertices of weight less than  $b$  are covered in  $G$ . By relabeling the vertices of  $G$  if needed we may assume that  $x_1 \geq \dots \geq x_n > 0$  and that  $G$  is left-compressed relative to the natural order on  $[n]$ .

We use induction on  $t$ . For the basis step, let  $t = 3$ . If  $x_{n-1} = b$ , then by Lemma 4.15,

$$\lambda(G, \vec{x}) \leq b(1 - 3b + 4b^2) \leq b(1 - 3b + 6b^2).$$

Hence, we may assume that  $x_{n-1}, x_n < b$ . By our assumption,  $\{n-1, n\}$  is covered in  $G$ . Since  $G$  is left-compressed, we have  $\forall 2 \leq i < j \leq n, 1ij \in G$ . Let  $G' = G[\{2, \dots, n\}]$ . By Lemma 4.14,  $G'$  is  $M_2^3$ -free. Since  $x_2 + \dots + x_n = 1 - x_1 = 1 - b$ ,  $\vec{y} = \frac{1}{1-b}(x_2, \dots, x_n)$  is a  $(\frac{b}{1-b})$ -bounded feasible weight vector on  $G'$ . Let  $b' = \frac{b}{1-b}$ . Since  $b \leq \frac{1}{8}$ ,  $b' = \frac{b}{1-b} \leq \frac{1}{7}$ . Since  $G'$  is  $M_2^3$ -free, and  $\vec{y}$  is a  $b'$ -bounded feasible weight vector on  $G'$ , by Lemma 4.12,

$$\lambda(G', \vec{y}) \leq (1-b)^3 \lambda(G', \vec{y}) \leq (1-b)^3 \cdot \frac{1}{2}b'(1-b')^2 = \frac{1}{2}(1-b)^3 \frac{b}{1-b} \left(\frac{1-2b}{1-b}\right)^2 = \frac{1}{2}b(1-2b)^2.$$

Since the total contribution to  $\lambda(G, \vec{x})$  from the edges containing 1 is at most  $\frac{1}{2}b(1-b)^2$ , we have

$$\lambda(G, \vec{x}) \leq \frac{1}{2}b(1-b)^2 + \frac{1}{2}b(1-2b)^2 = \frac{1}{2}b(2-6b+5b^2) < b(1-3b+6b^2).$$

Hence the claim holds. For the induction step, let  $t \geq 4$ . As before, if  $x_{n-1} = b$ , then by Lemma 4.15,

$$\lambda(G, \vec{x}) \leq \frac{t-1}{2}b(1-3b+4b^2) \leq \frac{t-1}{2}b(1-3b+6b^2).$$

Hence, we may assume that  $x_{n-1}, x_n < b$ . By our assumption,  $\{n-1, n\}$  is covered in  $G$ . Since  $G$  is left-compressed we have  $\forall 2 \leq i < j \leq n, 1ij \in G$ . By Lemma 4.14,  $G' = G[\{2, \dots, n\}]$  is  $M_{t-1}^3$ -free. Since  $\vec{y} = \frac{1}{1-b}(x_2, \dots, x_n)$  is a  $(\frac{b}{1-b})$ -bounded feasible weight vector on  $G'$ , by induction hypothesis,

$$\lambda(G', \vec{x}) = (1-b)^3 \lambda(G', \vec{y}) \leq (1-b)^3 \frac{t-2}{2} \frac{b}{1-b} \left( 1 - 3 \frac{b}{1-b} + 6 \left( \frac{b}{1-b} \right)^2 \right) = \frac{t-2}{2} b(1-5b+10b^2).$$

Since the total contribution to  $\lambda(G, \vec{x})$  from the edges containing 1 is at most  $\frac{1}{2}b(1-b)^2$ , we have

$$\begin{aligned} \lambda(G, \vec{x}) &\leq \frac{1}{2}b(1-b)^2 + \frac{t-2}{2}b(1-5b+10b^2). \\ &= \frac{1}{2}b[(t-1) - (5t-8)b + (10t-19)b^2] \\ &< \frac{t-1}{2}b(1-3b+6b^2), \end{aligned}$$

where the last inequality can be verified using the condition that  $0 < b \leq \frac{1}{3t-1}$  and  $t \geq 4$ . ■

**Theorem 4.17** *Let  $t \geq 2$  be an integer. There exists a positive real  $c_3 = c_3(t)$  such that the following holds. If  $G$  is an  $L_t^4$ -free 4-graph then  $\lambda(G) \leq \lambda(K_{3t}^4) = \frac{(3t-1)(3t-2)(3t-3)}{24(3t)^3}$ . Furthermore, if  $G$  also covers pairs and  $G \neq K_{3t}^4$ , then  $\lambda(G) \leq \lambda(K_{3t}^4) - c_3$ .*

*Proof.* Since we may consider a dense subgraph covering pairs, it suffices to prove the second statement. Suppose that  $G$  is on  $[n]$ . If  $n \leq 3t$ , then the result holds obviously since  $G \neq K_{3t}^4$ . Now suppose that  $n \geq 3t+1$ . Let  $\vec{x}$  be an optimum weight vector on  $G$ . Without loss of generality, suppose that  $x_1 = \max\{x_i : i \in [n]\}$ . Let  $a = x_1$ . By Fact 2.11, we have  $\lambda(G) = \frac{1}{4} \frac{\partial \lambda}{\partial x_1}$ . So it suffices to prove that  $\frac{\partial \lambda}{\partial x_1} \leq \frac{(3t-1)(3t-2)(3t-3)}{6 \cdot (3t)^3} - c_3$  for some positive real  $c_3 = c_3(t)$ . Since  $G$  is  $L_t^4$ -free,  $L(1)$  is an  $M_t^3$ -free 3-graph. Since  $G$  covers pairs,  $L(1)$  is a 3-graph on  $[n] \setminus \{1\}$  that contains no isolated vertex. Since  $G$  covers pairs and  $n \geq 3t+1$ , by Lemma 4.2,  $K_{3t-1}^3 \not\subseteq L(1)$ . Let  $\vec{y} = \frac{1}{1-a}(x_2, \dots, x_n)$ . Then  $\vec{y}$  is an  $(\frac{a}{1-a})$ -bounded feasible weight vector on  $L(1)$ . We consider two cases.

**Case 1.**  $a \geq \frac{1}{3t}$ .

Since  $L(1)$  is  $M_t^3$ -free,  $L(1) \neq K_{3t-1}^3$  and has no isolated vertex, by Theorem 3.3,

$$\lambda(L(1), \vec{y}) \leq \lambda(K_{3t-1}^3) - c_1 = \frac{(3t-1)(3t-2)(3t-3)}{6(3t-1)^3} - c_1.$$

Hence the claim holds by setting  $c_3 = c_1$ .

**Case 2.**  $a < \frac{1}{3t}$ .

Let  $b = \frac{a}{1-a}$ . Then  $b < \frac{1}{3t-1}$ . By Lemma 4.16, we have

$$\frac{\partial \lambda}{\partial x_1} = (1-a)^3 \lambda(L(1), \vec{y}) \leq (1-a)^3 \frac{t-1}{2} b(1-3b+6b^2).$$

Substituting in  $b = \frac{a}{1-a}$  and simplifying we get

$$\frac{\partial \lambda}{\partial x_1} \leq (t-1)a(5a^2 - \frac{5}{2}a + \frac{1}{2}).$$

Let  $f(a) = 5a^3 - \frac{5}{2}a^2 + \frac{1}{2}a$ . Note that  $f'(a) > 0$  always. So  $f(a)$  is increasing. Since  $a < \frac{1}{3t}$ , we have  $\frac{\partial \lambda}{\partial x_1} \leq (t-1)f(\frac{1}{3t}) = \frac{(t-1)(9t^2 - 15t + 10)}{2(3t)^3} < \frac{(t-1)(3t-1)(3t-2)}{2 \cdot (3t)^3} - c_3 = \frac{(3t-1)(3t-2)(3t-3)}{6(3t)^3} - c_3$ , for  $t \geq 2$  and sufficiently small positive real  $c_3 = c_3(t)$ . ■

**Corollary 4.18**  $\pi_\lambda(L_t^4) = 3!\lambda(K_{3t}^4) = \frac{[3t]_3}{(3t)^3}$ . ■

By Theorem 4.17, Corollary 4.18, Corollary 3.10 and Theorem 3.7, we get the following result.

**Theorem 4.19** *Let  $t \geq 2$  be an integer. Then  $ex(n, H_{3t}^{L_t^4}) = t_{3t}^4(n)$  for sufficiently large  $n$ . Moreover, if  $n$  is sufficiently large and  $G$  is an  $H_{3t}^{L_t^4}$ -free 4-graph on  $n$  vertices with  $|G| = t_{3t}^4(n)$  edges, then  $G = T_{3t}^4(n)$ .* ■

## 5 Concluding remarks

Another natural way to extend the Hefetz-Keevash result in [7] is to establish the maximum Lagrangian of an  $r$ -uniform intersecting family for  $r \geq 4$ , i.e. to determine the Lagrangian density of  $M_2^r$ , for  $r \geq 4$ . The situation there is quite different from the  $r = 3$  case. Hefetz and Keevash [7] conjectured that the maximum Lagrangian of an  $r$ -uniform intersecting family is achieved by a feasible weight vector on the star  $\{1ij : 2 \leq i < j \leq n\}$ . This conjecture was recently confirmed for all  $r \geq 4$  by Norin, Watts, and Yepremyan [13], who determined the Lagrangian density of  $M_2^r$  as well as the stability of the related Turán problem. For the stability part of their result, see also [22]. Independently, Wu, Peng, and Chen [21] had also confirmed the Hefetz-Keevash conjecture for  $r = 4$ .

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